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The Coordinate Group Symmetries of General Relativity*

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1. *Introduction*

The symmetry group of special relativistic theories, the Poincaré group, was imposed on physical theories to mirror the symmetries of the laws of nature under point mappings of the presumed Minkowskian space-time, thought to be the arena of physics. With the advent of the general theory of relativity the equations of the gravitational field were constructed so as to be invariant under arbitrary curvilinear point transformations of the space-time, now taken to be a four-dimensional pseudo-Riemannian manifold. Although the dynamical laws of all general relativistic theories are taken to have this enlarged symmetry group, the geometry of any particular space-time on which all the fields are defined no longer has this symmetry. In fact, in order to facilitate handling of the field equations of general relativity, it is often convenient to exploit the lack of symmetry of generic space-times to impose coordinate conditions upon the field variables, the metric tensor of the space-time.

The coordinate transformations leading to the preferred frames of reference in which the coordinate conditions are satisfied, or which preserve those conditions, in so far as they involve specific reference to the metric of the space-time, are best understood, not so much as point mappings within a given four dimensional space-time, but rather as mappings within the function space of the field variables of the theory, $g_{\mu\nu}(x^\alpha)$. (Greek indices are taken to range from 0 to 3, while Latin indices range from 1 to 3.) The general theory of relativity is thus seen to have a much larger natural symmetry group than was initially contemplated, namely transformations of the form

$$\bar{x}^\alpha = f^\alpha(x^\beta, g_{\mu\nu}(x^\alpha)) \tag{1.1}$$

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with $\bar{x}^\alpha$ considered functionals of the whole metric field $g_{\mu\nu}(x^\alpha)$. The space-time point mappings, transformations of the form

$$\bar{x}^\alpha = f^\alpha(x^\beta) \tag{1.2}$$

evidently form a subgroup (although not a normal subgroup).

In the pursuit of a quantum theory of gravitation many researchers have attempted to cast the equations of general relativity into a Hamiltonian form. The Hamiltonian of the theory would have to propagate the initial data forward in the (arbitrarily chosen) 'time' direction, and in general provide a realization of the transformation group of the theory. In contrast to all other physical theories, the dynamical laws of general-relativistic theories cannot be separated in any satisfactory manner from the symmetry group. The groups of either equations (1.1) or (1.2) have proved to be particularly intractable to such an approach. Employing a radically new approach Dirac (1958, 1959) finally succeeded in constructing a Hamiltonian version of the general theory of relativity, wherein both the evolution of the solution throughout the space-time and its symmetry group are realized by the unfolding of a canonical transformation in the phase space of the theory, generated by Hamiltonian functionals. However, this group is isomophic neither to that of equations (1.1) or (1.2). A principal purpose of this paper is to elucidate the relationship between these three groups associated with the general theory of relativity.

In Dirac's formulation of the general theory of relativity, infinitesimal coordinate transformations are generated by integrals over a space-like three-dimensional surface, whose integrands are the 'Hamiltonian constraints' $\mathscr{H}_s$, $\mathscr{H}_L$, multiplied by coefficients, the so-called descriptors, which represent, respectively, the magnitude of the time-like displacement normal to the space-like surface, and the infinitesimal coordinate transformations within the surface itself. The commutator algebra of the infinitesimal group of coordinate transformations is obtained simply by means of the Poisson brackets between the generators, and these are themselves generators of infinitesimal coordinate transformations. Remarkably, the descriptors of the commutator are independent of the derivatives of the original descriptors normal to the surface; otherwise the canonical procedure would fail. On the other hand, it is well known that the Lie derivative of the displacement vector, representing an infinitesimal coordinate transformation (or an infinitesimal mapping) with respect to another displacement vector, in general involves derivatives off the initial hypersurface, a fact that appears to foreclose the representation of the commutator algebra in terms of functions, or functionals, restricted to a three-dimensional domain. This apparent paradox will be resolved in this paper.

The need to clarify the relationship between the above three symmetry groups arises in part from the effort to find the observables of gravitation theory, that is, the functionals of the field variables that are invariant under space-time mappings. In our view such observables are required if one is to construct a quantum theory of gravitation. As there are three distinct symmetry groups that might be realized by space-time mappings, it would appear that we could have three different criteria for selecting observables ('gauge-invariant' variables), namely those functionals of the field variables that are invariant under the three groups, respectively. We shall show that all three groups act transitively within a given four-dimensional space-time manifold. (For the groups defined by equations (1.1) and (1.2) this fact is obvious. The principal work will be to ascertain this fact for the canonical group of Dirac.) As a consequence of this result, it follows that the invariants of the above three groups will, in the intended realizations of these groups as transformation groups of a four-dimensional Riemann manifold, yield equivalent sets of observables.

In addition this paper will make precise the assertion that in general relativity the identity of a world point is not preserved under the theory's widest invariance group. This assertion forms the basis for the conjecture that some physical theory of the future may teach us how to dispense with world points as the ultimate constituents of space-time altogether. A similar conjecture has been made, independently, by R. Penrose (1968).

2. *The Infinitesimal Groups*

Under the action of an infinitesimal coordinate transformation by a vector field ξ^ρ

$$\delta x^\rho \equiv \bar{x}^\rho - x^\rho = \xi^\rho \tag{2.1}$$

the metric tensor, $g_{\mu\nu}(x^\rho)$, transforms so that its functional dependence on the coordinates changes as follows

$$\delta g_{\mu\nu} \equiv \bar{g}_{\mu\nu}(x^\rho) - g_{\mu\nu}(x^\rho) = -(\xi_{\mu;\nu} + \xi_{\nu;\mu}) \tag{2.2}$$

The infinitesimal descriptor ξ^ρ for the transformations of the function space, equation (1.1), is an arbitrary functional of the metric (as well as a function of the coordinate); i.e.

$$\xi^\rho = \xi^\rho(x^\alpha, g_{\mu\nu}(\bar{x}^\rho) \tag{2.3}$$

The infinitesimal descriptor for a member of the subgroup of the space-time point mappings, equation (1.2), is not a functional of the metric tensor, but only a function of x^α,

$$\xi^\rho = \xi^\rho(x^\alpha) \tag{2.4}$$

The distinction between these two situations becomes critical when we seek to determine the commutator of two transformations described, respectively, by $\xi_1{}^\rho$ and $\xi_2{}^\rho$. For the familiar group of space-time transformations, that is for descriptors of the form equation (2.4), the descriptor of the commutator is the usual Lie derivative of the two vector fields. That is

$$\xi_c{}^\rho = \xi_1{}^\rho{}_{,\alpha}\xi_2{}^\alpha - \xi_2{}^\rho{}_{,\alpha}\xi_1{}^\alpha \tag{2.5}$$

In the more general case the commutator must be determined via the more involved expressions obtained by performing the indicated transformations in the function space. Thus

$$\delta g_{\mu\nu}(x) = \int d^4x' \left[\frac{\mathscr{D}\delta_1 g_{\mu\nu}(x)}{\mathscr{D}g_{\alpha\beta}(x')} \delta_2 g_{\alpha\beta}(x') - \frac{\mathscr{D}\delta_2 g_{\mu\nu}(x)}{\mathscr{D}g_{\alpha\beta}(x')} \delta_1 g_{\alpha\beta}(x') \right] \quad (2.6)$$

where $\delta_i g_{\mu\nu}$ denotes the set of expressions obtained from equation (2.2) by employing the relevant descriptor, $\xi_i{}^\rho$. For the special case of equation (2.4), equation (2.6) reduces to a relation equivalent to equation (2.5).

Dirac's canonical version of general relativity employs as the configuration variables the spatial portion of the metric, determined on a $x^0 = \text{constant}$ space-like hypersurface, $g_{mn}(x^s)$. The canonically conjugate momenta are found to be

$$p^{mn}(x^s) = \tfrac{1}{2}|g_{ab}|^{1/2}(g^{00})^{1/2}(e^{mr}e^{ns} - e^{mn}e^{rs})(g_{rs,0} - g_{0r}|_s - g_{0s}|_r) \quad (2.7)$$

where

$$e^{mn} \equiv g^{mn} - \frac{g^{0m}g^{0n}}{g^{00}} \quad (2.8)$$

is the reciprocal of the spatial metric g_{mn}, with respect to which all indices are raised and lowered; the vertical bar subscript denotes covariant differentiation with respect to that three-metric. The canonical momenta are not independent of the metric and of each other. They are related by the four constraints

$$\mathscr{H}_s(x) \equiv -2p_s^m(x)_m = 0 \quad (2.9)$$

$$\mathscr{H}_L(x) \equiv |g_{mn}|^{-1/2}(g_{mr}(x)g_{ns}(x) - \tfrac{1}{2}g_{mn}(x)g_{rs}(x))p^{mn}(x)p^{rs}(x) + \\ + |g_{mn}(x)|^{1/2}\,{}^3R(x) = 0 \quad (2.10)$$

where ${}^3R$ is the three-dimensional Ricci scalar formed from the g_{mn}.

These constraints now form the generator of the infinitesimal canonical transformation equivalent to the infinitesimal coordinate transformation, equation (2.1), on the phase space; thus

$$H(\xi) \equiv \int (\bar{\xi}^s \mathscr{H}_x + \bar{\xi}^L \mathscr{H}_L)\, d^3x \quad (2.11)$$

where

$$\xi^\rho \equiv \bar{\xi}^s \delta_s{}^\rho + \bar{\xi}^L \frac{g^{0\rho}}{\sqrt{(g^{00})}} \quad (2.12)$$

or, equivalently,

$$\bar{\xi}^L = \frac{1}{\sqrt{(g^{00})}}\xi^0$$

$$\bar{\xi}^s = \xi^s - \frac{g^{0s}}{g^{00}}\xi^0 \quad (2.13)$$

In order to obtain the Hamiltonian of the theory, or equivalently the generator of transformations in the 'time' direction, the required descriptor is $\xi^\rho = \delta_0{}^\rho$, or equivalently, from equation (2.13),

$$\bar{\xi}^L = \frac{1}{\sqrt{(g^{00})}}$$
$$\bar{\xi}^s = -\frac{g^{0s}}{g^{00}} = +g_{0m}\,e^{ms} \tag{2.14}$$

These expressions are to be inserted into equation (2.11).

Dirac's constraints, equations (2.9) and (2.10), are first-class. That is, their Poisson brackets can be expressed as linear combinations of the constraints themselves. The question arises whether their commutator algebra corresponds to some group of interest, say some subgroup of the group of curvilinear coordinate transformations. As stated in this vague fashion, it is of course not a well posed problem, for at this point we have neither made clear whether and in what fashion we intend to regard descriptors as functions of the dynamical variables, nor have we specified whether by the group of curvilinear coordinate transformations we are to understand the group of point mappings in the space-time, described by equation (1.2), or the function space mappings, described by equation (1.1).

In equation (2.11), let us regard the coefficients $\bar{\xi}^s$ and $\bar{\xi}^L$ as arbitrary functionals of the canonical variables; thus

$$\bar{\xi}^s(x^a) \equiv \bar{\xi}^s(x^a, g_{mn}, p^{rs})$$
$$\bar{\xi}^L(x^a) \equiv \bar{\xi}^L(x^a, g_{mn}, p^{rs}) \tag{2.15}$$

(It follows from equation (2.12) that the dynamical variables enter into the descriptors ξ^ρ in a somewhat restricted fashion.) Similarly let us define two further sets of descriptors η^ρ and μ^ρ. Then we assert

$$[H(\xi), H(\eta)] = H(\mu) \tag{2.16}$$

where $\bar{\mu}^s$, $\bar{\mu}^L$ are related to the $\bar{\xi}$- and $\bar{\eta}$-functionals thus:

$$\begin{aligned}\bar{\mu}^s(x) = {} & \bar{\eta}^s(x)_{,t}\,\bar{\xi}^t(x) - \bar{\xi}^s(x)_{,t}\,\bar{\eta}^t(x) + e^{st}(x)\,(\bar{\eta}^L(x)\bar{\xi}^L(x)_{,t} - \bar{\xi}^L(x)\,\bar{\eta}^L(x)_{,t}) + \\ & + \int d^3y(\bar{\eta}^t(y)\,[\bar{\xi}^s(x), \mathscr{H}_t(y)] + \bar{\eta}^L(y)\,[\bar{\xi}^s(x), \mathscr{H}_L(y)] - \\ & - \bar{\xi}^t(y)\,[\bar{\eta}^s(x), \mathscr{H}_t(y)] - \bar{\xi}^L(y)\,[\bar{\eta}^s(x), \mathscr{H}_L(y)])\end{aligned} \tag{2.17a}$$

$$\begin{aligned}\bar{\mu}^L(x) = {} & \bar{\xi}^s(x)\,\bar{\eta}^L(x)_{,s} - \bar{\eta}^s(x)\,\bar{\xi}^L(x)_{,s} + \int d^3y\,(\bar{\eta}^t(y)\,[\bar{\xi}^L(x), \mathscr{H}_t(y)] + \\ & + \bar{\eta}^L(y)\,[\bar{\xi}^L(x), \mathscr{H}_L(y)] - \bar{\xi}^t(y)\,[\bar{\eta}^L(x), \mathscr{H}_t(y)] - \\ & - \bar{\xi}^L(y)\,[\bar{\eta}^L(x), \mathscr{H}_L(y)])\end{aligned} \tag{2.17b}$$

The proof of this result merely requires straight-forward but tedious computations. The only point worth mentioning is that in the course of the computation one discards terms proportional to the square of the constraints. This is because of the manifold of solutions of the constraint equations such terms cannot alter the algebra. (We need not check for Jacobi identities, for these follow automatically for Poisson brackets.)

We have therefore shown that the phase space functionals $H(\xi)$ generate the Lie algebra of a subgroup of the full function-space group of coordinate transformations. In a later section we shall characterize this group in a more geometric fashion. Our conclusion will be that it is indeed a proper subgroup of the function space group (although not a normal subgroup).

By inspection of equation (2.17a) it is clear that the descriptors for which $\bar{\xi}^s(x)$, $\bar{\xi}^L(x)$ are independent of the canonical variables do *not* form a closed subalgebra of the above subgroup.

It remains to investigate whether the subgroup of space-time point mappings, equation (1.2), can be recovered as a subgroup of the phase-space group generated by $H(\xi)$. We shall now show that this is *not* the case. (In view of the fact that the space-time point mappings evidently form a subgroup of the group of function space mappings, it will immediately provide an alternative proof that the phase-space group is a *proper* subgroup of the function space group.)

The Lie algebra of the group of point mappings in space-time is characterized by those descriptors, ξ^ρ, which are independent of the dynamical variables. Thus for the point mappings, equation (2.4), $\bar{\xi}^s$ and $\bar{\xi}^L$ can depend on the canonical variables alone only if the $g^{\rho 0}$ in turn depend on the canonical variables [cf. equation (2.13)]. As the components $g^{\rho 0}$ are not themselves canonical variables, to make them depend on the canonical variables amounts to the imposition of a species of coordinate conditions. Hence, it is appropriate to inquire whether coordinate conditions of the form

$$g^{\rho 0}(x) \equiv g^{\rho 0}(x, g_{mn}, p^{rs}) \tag{2.18}$$

can be found for which descriptors of the type equation (2.13) form a closed algebra such that (a) the property that ξ^ρ does not depend on the canonical variables is preserved, and that (b) the Lie algebra of the transformations generated is consistent with the Lie derivative of the descriptors, equation (2.5).

Substituting relations for ξ^ρ and η^ρ corresponding to equation (2.13) into equations (2.17) and equating to zero all terms that would compel μ^ρ to depend on the canonical variables, leads after much computation, to the result that this property of the descriptors will be preserved provided $g^{\rho 0}$ satisfies the following commutation relations:

$$\begin{aligned}\left[\frac{g^{0s}(x)}{g^{00}(x)}, H_t(y)\right] = {} & \left(\frac{g^{0s}{}_{,t}}{g^{00}} - \frac{g^{00}{}_{,t}}{g^{00}}\frac{g^{0s}}{g^{00}}\right)\delta(x-y) + a_t{}^s(x)\,\delta(x-y) - \\ & - \frac{g^{0s}(x)}{g^{00}(x)}\frac{\partial}{\partial x^t}\delta(x-y) - \delta_t{}^s\frac{g^{0m}(x)}{g^{00}(x)}\frac{\partial}{\partial x^m}\delta(x-y) + \\ & + d_t^{sm}(x)\frac{\partial}{\partial x^m}\delta(x-y) + f_t^{sa\cdots b}(y)\frac{\partial}{\partial x^a}\cdots\frac{\partial}{\partial x^b}\delta(x-y)\end{aligned} \tag{2.19a}$$

$$\left[\frac{g^{00}(x)}{g^{00}(x)}, H_L(y)\right] = k^s(x)\,\delta(x-y) - \frac{g^{st}(x)}{\sqrt{[g^{00}(x)]}}\frac{\partial}{\partial x^t}\delta(x-y) +$$
$$+ \sqrt{[g^{00}(x)]}\,c^{st}(x)\frac{\partial}{\partial x^t}\delta(x-y) +$$
$$+ d_m^{st}(x)\frac{g^{0m}(x)}{\sqrt{[g^{00}(x)]}}\frac{\partial}{\partial x^t}\delta(x-y) +$$
$$+ \left(f_t^{sa\ldots b}(y)\frac{g^{0t}(y)}{\sqrt{[g^{00}(y)]}} + \sqrt{[g^{00}(y)]}\,h^{sa\ldots b}(y)\right) \times$$
$$\times \frac{\partial}{\partial x^a}\cdots\frac{\partial}{\partial x^b}\delta(x-y) \qquad (2.19b)$$

$$\{\sqrt{[g^{00}(x)]}, H_t(y)\} = \{\sqrt{[g^{00}(x)]}\}_{,t}\,\delta(x-y) - a_t(x)\sqrt{[g^{00}(x)]}\,\delta(x-y) -$$
$$- b_t{}^r(x)\sqrt{[g^{00}(x)]}\frac{\partial}{\partial x^r}\delta(x-y) \qquad (2.19c)$$

$$\{\sqrt{[g^{00}(x)]}, H_L(y)\} = q(x)\,\delta(x-y) + g^{0m}(x)\,(\delta_m{}^m + b_m{}^m(x))\frac{\partial}{\partial x^m}\delta(x-y) -$$
$$- g^{00}(x)\,l^m(x)\frac{\partial}{\partial x^m}\delta(x-y) \qquad (2.19d)$$

where the various functions denoted by $a_t(x)$, $a_t{}^s(x)$, $b_t{}^s(x)$, etc. are arbitrary functions of position, but are not to depend on the dynamical variables. We have not succeeded in identifying the full range of functionals $g^{\rho 0}$ of the form (2.18) that satisfy these commutation relations. We can, however, obtain a partial result by examining the descriptors of the commutator, assuming equations (2.19):

$$\mu^s = \xi^t\eta^s{}_{,t} - \eta^t\xi^s{}_{,t} + a_t{}^s(\xi^t\eta^0 - \eta^t\xi^0) + c^{st}(\xi^0{}_{,t}\eta^0 - \eta^0{}_{,t}\xi^0) +$$
$$+ d_t^{sm}(\xi^t{}_{,m}\eta^0 - \eta^t{}_{,m}\xi^0) + h^{sa\ldots b}(\xi^0\eta^0{}_{,a\ldots b} - \eta^0\xi^0{}_{,a\ldots b}) \qquad (2.20a)$$
$$\mu^0 = \xi^s\eta^0{}_{,s} - \eta^s\xi^0{}_{,s} + a_t(\eta^t\xi^0 - \xi^t\eta^0) + b_t{}^r(\eta^t{}_{,r}\xi^0 - \xi^t{}_{,r}\eta^0) +$$
$$+ l^r(\xi^0{}_{,r}\eta^0 - \eta^0{}_{,r}\xi^0) \qquad (2.20b)$$

By inspection μ^ρ is indeed independent of the dynamical variables whenever ξ^ρ and η^ρ are. But in general μ^ρ is not the Lie derivative of ξ^ρ and η^ρ. Only for the subgroup of the spatial coordinate transformation, i.e. $\xi^0 = \eta^0 = 0$, are the commutation relations equations (2.20) completely equivalent to the Lie derivatives of the descriptor fields. [Thus the spatial point mappings form a subgroup of the group of phase space mappings. In fact it is the subgroup generated by the first three constraints, equation (2.9).] If we require the expressions of equations (2.20) to be Lie derivatives under more general circumstances, this can be accomplished only by requiring the descriptor fields to be the solution of the following equations [obtained by equating the coefficients of η^0 in equations (2.20) to $\xi^{\rho}{}_{,0}$]

$$\xi^s{}_{,0} = a_t{}^s \, \xi^t + {}^{st}_{c} \, \xi^0{}_{,t} + d_t^{sm} \, \xi_{,m} - -h^{sa \ldots b} \, \xi^0{}_{,a \ldots b} \tag{2.21a}$$

$$\xi^0{}_{,0} = -a_t \, \xi^t - b_t{}^r \, \xi^t{}_{,r} + l^r \, \xi^0{}_{,r} \tag{2.21b}$$

These represent a set of equations of the first differential order in time (x^0) for the propagation of descriptors off the initial space-like hypersurface. Because the transformation laws for the metric components $g^{0\mu}$ involve time derivatives of the descriptors, there are no restrictions on the choice of descriptors on the initial hypersurface. As the descriptors propagate, so do the coordinate conditions (2.18); no additional restrictions are called for. But the propagation conditions (2.21) are most difficult to evaluate. Apart from the question of whether coefficients a_t, $a_t{}^s$, $b_t{}^s$, etc. can be found such that for fixed assignment of them equations (2.21) can be preserved under Lie derivation, it is quite clear that descriptor fields determined by such first-order equations are much too restricted to contain the full freedom available in the group of space-time point mappings, equations (1.2). Thus that group is evidently *not* a subgroup of the group of phase-space mappings generated by the Dirac constraints.

In the next section infinitesimal transformations of the general type (2.3) will be examined. For this purpose the canonical formalism is inappropriate; the phase space $g_{mn}(x' \ldots x^3)$, $p^{mn}(x' \ldots x^3)$ will be replaced by the larger function space $g_{\mu\nu}(x', \ldots x^3, x^0)$.

3. *Q-Type Coordinate Transformations*

Transformation groups are basically permutations. A set of elements is mapped on itself bi-uniquely; the mappings if appropriately selected form a group. In mappings described by equations (2.4) (hereafter referred to as C-type) the elements are simply the points of space-time, and the mappings are selected to be diffeomorphic. C-type coordinate transformations will, incidentally, also map the function space of metric fields on itself, but this representation of the group of C-type transformations is relatively cumbersome. Each member of the group is, of course, a precise prescription that associates with every element P its map P' regardless of any other information that may be available (such as the metric field).

The more general transformations described by equation (2.3), (hereafter referred to as Q-type), map points on points, to be sure, but the prescription associates with a given P a map P' that also depends on the metric field. Hence the set of elements that is being mapped on itself uniquely and without adornments is the function space of metric fields. The mappings of that function space on itself induced by Q-type coordinate transformations form a transformation group. If equations (2.2) and (2.6) are combined to derive the descriptor of an infinitesimal Q-type transformation from the descriptors of the two constituent transformations, one obtains the following rule:

$$\xi_c{}^{\mu}(x) = \xi^{\mu}_{1,\rho}\,\xi_2{}^{\rho} - \xi^{\mu}_{2,\rho}\,\xi_1{}^{\rho} - \int d^4x \left\{ \frac{\mathscr{D}\xi_1{}^{\mu}(x)}{\mathscr{D}_{g\alpha\beta}(x')} [\xi_{2\alpha}(x')_{;\beta} + \xi_{2\beta}(x')_{;\alpha}] - \right.$$
$$\left. - \frac{\mathscr{D}\xi_2{}^{\mu}(x)}{\mathscr{D}_{g\alpha\beta}(x')} [\xi_{1\alpha}(x')_{;\beta} + \xi_{1\beta}(x')_{;\alpha}] \right\} \qquad (3.1)$$

In this formulation it is transparent that the commutator reduces to the one for C-type transformations if the constituent descriptors are in fact independent of the metric.

In case the dependence of the constituent descriptors on the metric is local, equation (3.1) is simplified considerably; if, for instance, the descriptors are functions of the undifferentiated local values of the $g_{\mu\nu}(x)$, then the expression (3.1) reduces to

$$\xi_c{}^{\mu}(x) = \xi^{\mu}_{1,\rho}\,\xi_2{}^{\rho} - \xi^{\mu}_{2,\rho}\,\xi_1{}^{\rho} - \frac{\partial\xi_1{}^{\mu}}{\partial g_{\alpha\beta}}(\xi_{2\alpha;\beta} + \xi_{2\beta;\alpha}) + \frac{\partial\xi_2{}^{\mu}}{\partial g_{\alpha\beta}}(\xi_{1\alpha;\beta} + \xi_{1\beta;\alpha}) \qquad (3.2)$$

The new descriptors are functions not only of the metric at the point with the coordinates (x), but of the first derivatives of the metric as well. For local dependence, the descriptors of the commutator will always involve derivatives of higher order than the descriptors of the constituent infinitesimal transformations; no local dependence on the metric and its derivatives to a finite order will form a group. That is why it is preferable to consider in Q-type transformations the primarily non-local (i.e. functional) dependence of the descriptors on the metric.

The set of all metric fields may be collected in equivalence classes, each of which represents one pseudo-Riemannian manifold in all conceivable coordinate systems. Under either C-type or Q-type transformations each equivalence class is mapped on itself; it is an invariant subspace of the function space of metric fields. Its mappings provide a realization of the group of coordinate transformations, which in some cases (the presence of isometries) is not faithful.

Among the pseudo-Riemannian manifolds there are those that are solutions of Einstein's vacuum field equations, the Ricci-flat manifolds. Without loss of physically valuable generality one may confine one's attention to representations involving Ricci-flat manifolds (generally without isometries).

A Ricci-flat manifold may be represented simply as a metric field that obeys Einstein's field equations. An alternative representation is in terms of Cauchy data; Dirac (1958, 1959) has cast these into a canonical form. A complete set of data consists of the field $g_{mn}(x^1, \ldots, x^3)$, $p^{mn}(x^1, \ldots, x^3)$ which must satisfy at each (three-) point the four Hamiltonian constraints,

$\mathscr{H}_L = 0$, $\mathscr{H}^s = 0$ [cf. equations (2.9), (2.10)]. Each such field uniquely determines a Ricci-flat manifold, but one-and-the-same Ricci-flat manifold may be represented by an infinity of such fields. Given one of them, equivalent fields may be obtained by changing the coordinatization of the same space-like hypersurface, by going over to another hypersurface, or by combining these two operations. Infinitesimal changes of this kind are generated by the Hamiltonian constraints themselves. Clearly, all possible canonical fields may be decomposed into invariant subspaces, or equivalence classes, each equivalence class consisting of all canonical fields representing the same Ricci-flat manifold. The canonical transformations corresponding to the replacement of one canonical field by another belonging to the same Ricci-flat manifold permit the construction of transformation groups. These transformation groups may be realized (not necessarily faithfully) by the transformations within one equivalence class.

Whereas the equivalence classes of four-dimensional fields and those of canonical fields are directly related to each other, the two kinds of representation of Ricci-flat manifolds, and the transformation groups connecting them, are not. One four-dimensional metric field may be regarded as a congruence of three-dimensional hypersurfaces, and each hypersurface will admit a different canonical field which on that hypersurface is adapted to the given four-dimensional metric field. Conversely, given a canonical field on a hypersurface, there is an infinity of possible four-dimensional coordinate systems, and hence of metric fields, corresponding to the one canonical field on that one hypersurface.

Consider the phase space of $g_{mn}(x)$, $p^{mn}(x)$, that is to say the function space of these fields, and imbedded in it the subspace of canonical fields satisfying all of Dirac's Hamiltonian constraints. In that subspace one canonical field is represented by a point; a four-dimensional metric field corresponds to a one-parametric curve, with the parameter x^0 specified along that curve. Whereas canonical transformations map point onto point, four-dimensional coordinate transformations map parametrized curve onto parametrized curve. Thus the constraint subspace of phase space is suitable for representing both four-dimensional Ricci-flat metric fields and three-dimensional canonical fields, and one can visualize the transformations appropriate to either.

Point-to-point mappings also map curves onto curves, and will even provide a mapping of the corresponding parameters, under the simple prescription that the mapping will transfer unchanged the value of x^0 from any point to its image. But curve-to-curve mappings are not necessarily point-to-point mappings. Within one equivalence class consider two parametrized curves that have one point in common (i.e. two metric fields that coincide on one common hypersurface, $x^0 =$ constant, and have their first off-surface derivatives in common as well). Their maps need not intersect anywhere: hence the original point of intersection has no map at all. A point-to-point mapping will map a given point (i.e. canonical field) on a specific point, without reference to a curve passing through it.

For further discussion of these canonical mappings, independent of their realization within the orthodox Hamiltonian formalism, the concept of D-invariance (Bergmann, 1962) is of considerable value. Introduced originally by Dirac (1958, 1959) (although not by name), D-invariance is a property possessed by some, but not all, components of geometric objects defined on the four-dimensional space-time manifold. The D-invariant components possess the property of remaining unchanged under all those four-dimensional coordinate transformations that reduce to the identity transformation on the initial space-like hypersurface on which the canonical Cauchy data are defined. For instance, the three spatial components of a covariant vector are D-invariant, as are the canonical fields g_{mn} and p^{mn}.

Canonical mappings whose descriptors ξ^s, ξ^L are D-invariant are themselves D-invariant, whereas those mappings corresponding to general four-dimensional coordinate transformations are not. This formulation recommends itself in that it is independent of the appearance and technology of canonical manipulations; the characteristic properties of canonical mappings can be examined with the help of the technique exemplified by equation (3.1), a technique equally applicable to the most general four-dimensional Q-type mappings.

A D-invariant mapping of a three-dimensional space-like hypersurface on a like hypersurface requires information only about the canonical field g_{mn}, p^{mn} on that hypersurface. This information really provides data on two disparate sets of circumstances: (1) It identifies the four-dimensional Ricci-flat manifold (equivalence class), and (2) it tells on which hypersurface we are and how that hypersurface is coordinated. By contrast, a general four-dimensional Q-type coordinate transformation requires knowledge of the metric field throughout the four-dimensional manifold. In addition to specifying the pseudo-Riemannian geometry, this information also particularizes the coordinate system chosen throughout that manifold.

From the foregoing it is clear that we have to deal with three distinct types of transformations. The general Q-type coordinate transformations form a group in that they map the function space $g_{\mu\nu}(x^0, \ldots, x^3)$ on itself in such a manner that the map of a world point is another world point, depending on the whole metric field of the manifold. These mappings are transitive only within each invariant subspace (an equivalence class). That these transformations form a group, or at least a groupoid, is seen from their definition. As a corollary, the infinitesimal mappings form a Lie algebra, which is described, e.g. by equation (3.1).

The C-type transformations form a subgroup of the first group, and the Lie algebra of infinitesimal C-type transformations is given by equation (2.5). The subgroup is not an invariant subgroup, but within an equivalence class (as described above) it is transitive; that is to say, there is a C-type transformation transforming any one particular coordinatization of a chosen Ricci-flat manifold. It is worth noting that, except for manifolds with isometries, there are no non-trivial C-number transformations that map a given metric field on itself; hence (again with the exception of isometries) given two coordinatizations there is precisely one C-type transformation mapping one on the other. By contrast, there are infinitely many such Q-type transformations.

The canonical coordinate transformations form another subgroup of the Q-type transformations. As a Ricci-flat manifold is identified by its Cauchy data, canonical coordinate transformations also map each equivalence class (of Cauchy data describing one-and-the-same Ricci-flat manifold) on itself. Infinitesimal canonical coordinate transformations may be characterized in terms of their canonical generators, which are linear combinations (i.e., three-dimensional integrals over linear combinations) of the Hamiltonian constraints, with the coefficients being D-invariant functionals or, equivalently, functionals of the canonical fields g_{mn}, p^{mn} only. The Lie algebra of the infinitesimal canonical coordinate transformations may be obtained by way of the Poisson brackets between their generators, or, alternatively, through equation (3.1). Relating the descriptors appearing in that formalism to the coefficients of the Hamiltonian constraints, $\bar{\xi}^s$, $\bar{\xi}^L$, by the expression (2.12), all four $\bar{\xi}$ must be D-invariant. The subgroup of canonical coordinate transformations again is not an invariant subgroup of the Q-type coordinate transformations. Within an equivalence class it is transitive, in that it is possible to map any set of canonical Cauchy data for a given Ricci-flat manifold on any other like set describing the same Ricci-flat manifold. For any given canonical field there are non-trivial transformations within the group that map the particular chosen canonical field on itself.

Both the Q-type group and the two subgroups considered here (the C-type group and the canonical group) have the same orbits, the equivalence classes. Hence an invariant with respect to any of the sub-groups must be a constant over the domain of each equivalence class, and therefore an invariant with respect to all three groups.

The Lie algebras of the two infinitesimal subgroups are subalgebras of the Lie algebra of the infinitesimal Q-type group, but neither is an invariant subalgebra. For proof it suffices to form the commutator between an infinitesimal C-number transformation and an infinitesimal canonical transformation, a straight-forward operation. In the generic case the commutator turns out to belong to neither of these two classes.

Even the C-type transformations may be considered mappings of function spaces on themselves (though these mappings surely are not the most convenient representations imaginable). With the C-type transformations the world points, and the values of the metric tensor there, represent entities that are mapped onto each other, intact. That is to say, under a C-type transformation the metric tensor at one world point determines, by itself, the metric tensor at the world point that is the image of the first one. If we consider the set of all metric fields a fiber bundle, the four-dimensional manifold its base, and the variety of all possible metric tensors at one world point a fiber, then the C-type transformations map each entire fiber on one entire fiber. In this sense the C-type transformations preserve the identities of distinct world points intact.

Q-type transformations and canonical coordinate transformations do not have that property. Consider one world point, with the variety of all possible metric tensors at that point. Under a Q-type transformation that world point is mapped on another world point, whose identity depends on the metric assumed. Given all possible metric fields, but only one Q-type transformation, the set of all images of one world point may be quite a large point set, a four dimensional variety in the most general case. Essentially the same statements hold for canonical coordinate transformations. As we are concerned with transformations that form groups, each transformation has its inverse; it follows that under Q-type transformations and under canonical coordinate transformations one world point may be the image of many world points, depending on the assumed metric fields. As world point mappings, these transformations are both one-to-many and many-to-one, hence non-unique in both directions.

Because of the equivalence of orbits under all three groups, the invariants of a general-relativistic theory are unaffected by the substitution of one group for the other. Thus we have here a set of theories with well-understood covariance properties, whose invariance group cannot be described as a group of transformations mapping a four-dimensional point set on itself.

4. *Concluding Remarks*

To summarize the results of this paper, its principal achievement consists of the determination of the mutual relationship of three distinct transformation groups, all, and indiscriminately, referred to as coordinate transformations, with respect to which general-relativistic theories are form-invariant. The most general of these transformation groups comprises all diffeomorphic mappings of a space-time manifold (whose fields obey the field equations) on itself, where the mapping will in general depend on the fields defined on the manifold. The other two types of mapping considered form subgroups of the first. One is the group of mappings independent of the fields; the other consists of mappings that depend on the fields only by way of Cauchy data on a hypersurface necessary to define a physical situation uniquely and irrespective of the particular four-dimensional coordinatization chosen for its description. The properties of the three distinctive Lie algebras belonging to these different groups serve to explain the apparent paradox inherent in the Poisson brackets discovered by Dirac, which are appropriate to the D-invariant mappings but not to the two other groups.

The analysis presented in this paper points to the possible formulation of realistic physical theories in which the world point plays no intrinsic role; in an appropriate realization such theories might be amenable to a form not altogether unlike theories well known today.

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